NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3832

ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

II - SIMILARITY WITH RESPECT TO STATIONARY POLAR COORDINATES

By Howard Z. Herzig and Arthur G. Hansen

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

Washington
November 1956

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3832

ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

II - SIMILARITY WITH RESPECT TO STATIONARY POLAR COORDINATES

By Howard Z. Herzig and Arthur G. Hansen

SUMMARY

Solutions of mainstream flow patterns for three-dimensional, laminar, incompressible thin-boundary-layer flows (over flat or slightly curved surfaces) having similarity with respect to stationary polar coordinates in the plane of the surface are derived. The solutions are summarized in a table.

INTRODUCTION

Considerable attention has been devoted in laminar-boundary-layer research to theoretical solutions of the two- and three-dimensional incompressible-boundary-layer equations using the "similarity" approach. In this method, the partial differential boundary-layer equations are transformed by means of a similarity parameter η and rewritten in terms of functions of η, their derivatives, the mainstream velocity components, and their derivatives. Solutions are then sought for the mainstream flow conditions for which the transformed equations reduce to ordinary differential equations for the functions of η (refs. 1 to 10). Some experimental evidence is presented in reference 10 in support of this kind of theoretical development for laminar flows. Reference 11 presents a systematic approach to similarity-type solutions using a generalized similarity parameter. As a result, reference 11 has obtained solutions for the permissible mainstream flows for all the boundary-layer flows having classical similarity with respect to stationary rectangular coordinates. The present report is an extension of the work of reference 11. Solutions are sought for the mainstream flows in stationary cylindrical coordinates for all the boundary-layer flows having classical similarity with respect to the polar coordinates in the plane of the surface.

SYMBOLS

a,b,c,C	constants
$F,F(\eta)$	function of similarity parameter, $u \equiv UF'(\eta)$
f	arbitrary function
$G,G(\eta)$	function of similarity parameter, $w \equiv WG'(\eta)$ for $W \neq 0$
$g,g(r,\theta)$	function of coordinates r and θ
k,m,n	constants
r,θ,y	polar coordinates
U,W	mainstream velocity components in θ,r directions, respectively
u,v,w	boundary-layer velocity components in θ,y,r directions, respectively
$\overline{W}$	function of coordinates r and θ, $w \equiv \overline{W}G'(\eta)$ for $W = 0$
η	similarity variable, $\eta = yg(r,\theta)/\sqrt{\nu}$
ν	coefficient of kinematic viscosity

Subscripts:

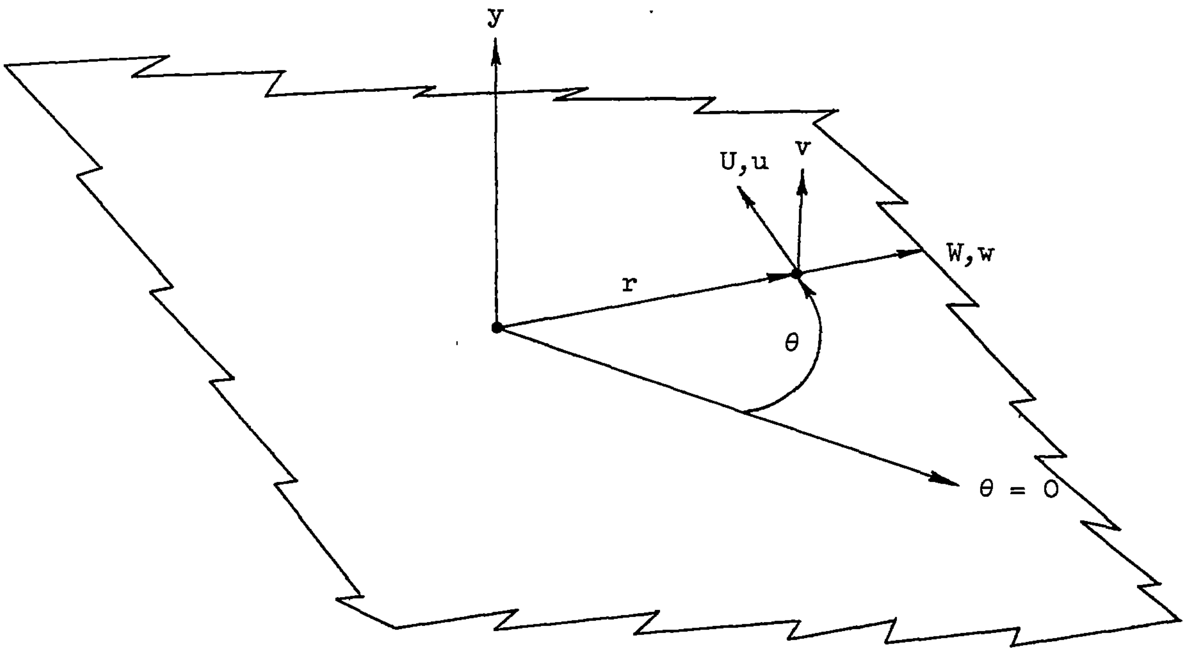
$1,2,3, \ldots$	index numbers

Primes denote differentiation.

ANALYSIS

Boundary-Layer Equations in Stationary Polar Coordinates

The three-dimensional laminar, incompressible, thin-boundary-layer equations in stationary cylindrical coordinate form for flows over flat (or slightly curved) surfaces with coordinate axes as shown in this sketch



are given by

$$\frac{uw}{r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = \frac{UW}{r} + \frac{U}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial r} \qquad (1a)$$

$$-\frac{u^2}{r} + \frac{u}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial y} - \nu\frac{\partial^2 w}{\partial y^2} = -\frac{U^2}{r} + \frac{U}{r}\frac{\partial W}{\partial \theta} + W\frac{\partial W}{\partial r} \qquad (1b)$$

where u, w, and v are the boundary-layer velocity components in the θ, r, and y directions, respectively.

Equations (1a) and (1b) are the boundary-layer flow equations in the tangential and radial directions, respectively. Consistent with the restriction to thin-boundary-layer flows for flat (or slightly curved) surfaces, the mainstream velocity components are

$$U = U(r,\theta)$$

$$W = W(r,\theta)$$

The continuity equation for the boundary-layer flow is given by

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} + \frac{w}{r} + \frac{\partial v}{\partial y} = 0 \tag{1c}$$

The boundary conditions are

$$\left.\begin{array}{l} u = w = v = 0 \quad \text{for} \quad y = 0 \\ \left.\begin{array}{l} u \to U \\ w \to W \end{array}\right\} \text{as} \quad y \to \infty \end{array}\right\} \tag{1d}$$

4177

Transformation of Equations Using Generalized η

The method of search for symmetric solutions described in reference 11 suggests exact solutions (i.e., solutions for flows having similarity with respect to the polar coordinates r,θ in the plane of the surface) of equations (1) can be obtained as a result of transformations to new coordinates r, θ, and η where the space variable η is of the form

$$\eta \equiv \frac{y}{\sqrt{\nu}}\, g(r,\theta) \tag{2}$$

In the rectangular coordinate systems, when either component of mainstream flow equals zero, straight mainstream flows result, and the boundary-layer flow is two-dimensional with no secondary-flow overturning. In polar coordinates when $U = 0$, the mainstream flows are straight and $u = 0$ (ref. 12). However, when $U \neq 0$, there is curvature of the flow streamlines even though $W = 0$, and three-dimensional boundary-layer overturning does result. In polar coordinates, as will be seen subsequently, it will be necessary to treat separately the cases of $W = 0$ and $W \neq 0$.

$\underline{W \neq 0}$. - Following the rectangular-coordinate-system analyses $\underline{(\text{ref. } 11)}$,

$$u \equiv UF'(\eta),\ U \neq 0 \tag{3a}$$

$$w \equiv WG'(\eta),\ W \neq 0 \tag{3b}$$

The conditions on F' and G' required to satisfy boundary conditions on u and w are

$$\left.\begin{aligned} F'(0) = G'(0) &= 0 \\ \lim_{\eta \to \infty} F'(\eta) &= 1 \\ \lim_{\eta \to \infty} G'(\eta) &= 1 \end{aligned}\right\} \tag{4a}$$

4177

Now v may be determined by integration of the continuity equation using equations (3):

$$v = \frac{-\sqrt{\nu}}{g}\left[\left(\frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}\right) F + \left(\frac{\partial W}{\partial r} - W\frac{\partial \ln g}{\partial r} + \frac{W}{r}\right) G\right] -$$

$$\frac{U}{r}\frac{\partial \ln g}{\partial \theta} yF' - W\frac{\partial \ln g}{\partial r} yG' + f(r,\theta) \tag{5}$$

In order that $v = 0$ for $y = 0$ as required, it is possible without loss of generality to set the boundary conditions

$$\left.\begin{aligned} F(0) = G(0) &= 0 \\ \text{and} \qquad f(r,\theta) &= 0 \end{aligned}\right\} \tag{4b}$$

(See appendix C of ref. 11 for a discussion of the necessary and sufficient boundary conditions.)

Upon substitution of equations (3) and (5), equation (1a) becomes ($W \neq 0$)

$$\frac{W}{r}(F'G' - GF'' - 1) + \frac{1}{r}\frac{\partial U}{\partial \theta}(F'^2 - FF'' - 1) + W\frac{\partial \ln U}{\partial r}(F'G' - 1) +$$

$$\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta} FF'' + \frac{W}{2}\frac{\partial \ln g^2}{\partial r} GF'' - \frac{\partial W}{\partial r} GF'' - g^2 F''' = 0 \tag{6}$$

and equation (1b) becomes

$$\frac{U^2}{Wr}(1 - F'^2) + \frac{U}{r}\frac{\partial \ln W}{\partial \theta}(F'G' - 1) + \frac{\partial W}{\partial r}(G'^2 - GG'' - 1) - \frac{1}{r}\frac{\partial U}{\partial \theta} FG'' +$$

$$\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta} FG'' + \frac{W}{2}\frac{\partial \ln g^2}{\partial r} GG'' - \frac{W}{r} GG'' - g^2 G''' = 0 \tag{7}$$

The purpose of this investigation is to determine mainstream flow solutions for which the transformed equations (6) and (7) reduce to ordinary differential equations. As in reference 11, the mainstream flow conditions that make the coefficient of the functions of η proportional are sought. Under these ordinary-differential-equation conditions (abbreviated to o.d.e. conditions), the common variable terms in the equations may be divided out, leaving ordinary differential equations for F and G. (The actual numerical solutions of the ordinary differential equations are not attempted herein.) Although coefficients of similar functions may be grouped and made proportional, the two techniques can be shown to be equivalent. 4177

For convenience, the coefficients for the functions of η in equations (6) and (7) are presented here in the order of their appearance. With $W \neq 0$, they are:

① $\frac{W}{r}$	⑥ $\frac{\partial W}{\partial r}$
② $\frac{1}{r}\frac{\partial U}{\partial \theta}$	⑦ g^2
③ $W\frac{\partial \ln U}{\partial r}$	⑧ $\frac{U^2}{Wr}$
④ $\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}$	⑨ $\frac{U}{r}\frac{\partial \ln W}{\partial \theta}$
⑤ $\frac{W}{2}\frac{\partial \ln g^2}{\partial r}$	

The o.d.e. conditions require these nine coefficients to be proportional to each other. The most general case, $W = W(r,\theta)$, can be solved readily. From o.d.e. conditions on ①, ⑦, and ⑧,

$$W = c_1 U \tag{8}$$

$$g^2 = \frac{c_2 U}{r} \tag{9}$$

Then from o.d.e. conditions on ② and ⑧, using equation (8),

$$\frac{\partial U}{\partial \theta} = \frac{c_3}{c_1} U \tag{10}$$

which may be integrated to give

$$U = e^{\frac{c_3}{c_1}\theta} f_1(r) \tag{11}$$

4177

From o.d.e. conditions on ① and ⑥, using equations (8) and (11),

$$\frac{f_1'(r)}{f_1(r)} = c_4 \frac{1}{r} \tag{12}$$

Therefore,

$$f_1(r) = c_5 r^{c_4} \tag{13}$$

Using equation (9) and redefining the constants involved for convenience permits W, U, and g^2 to be written

$$\left.\begin{aligned} W(r,\theta) &= br^n e^{m\theta} \\ U(r,\theta) &= ar^n e^{m\theta} \\ g^2(r,\theta) &= cr^{n-1} e^{m\theta} \end{aligned}\right\} \tag{14}$$

No further restrictions on the forms of W, U, and g^2 are required to satisfy the o.d.e. conditions for the remaining coefficients ③, ④, ⑤, and ⑨. Substitution of equations (8) and (9) indicates that ③ and ⑤ are already proportional to ⑥, while ④ and ⑨ are proportional to ②.

In addition, further analysis shows that the remaining possible main flows $W = W(r)$, $W = W(\theta)$, or $W = \text{constant} \neq 0$ can be obtained from equations (14) by suitable choices of n, m, a, and b. For example, when $m = 0$,

$$\left.\begin{aligned} W(r) &= br^n \\ U(r) &= ar^n \\ g^2(r) &= cr^{n-1} \end{aligned}\right\} \tag{15}$$

The ordinary differential equations for $W(r,\theta) \neq 0$ now can be obtained by substitution of equations (14) into equations (6) and (7);

$$b(n + 1)(F'G' - 1) + am\left(F'^2 - \frac{FF''}{2} - 1\right) - \frac{b(n + 3)}{2} GF'' - cF''' = 0 \tag{16}$$

$$\frac{a^2}{b}(1 - F'^2) + am\left(F'G' - \frac{FG''}{2} - 1\right) + bn(G'^2 - 1) - \frac{b(n+3)}{2} GG'' - cG''' = 0 \tag{17}$$

$\underline{W = 0}$. - When $W = 0$, the corresponding boundary-layer equations (1) become

$$\frac{uw}{r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = \frac{U}{r}\frac{\partial U}{\partial \theta} \tag{18a}$$

$$-\frac{u^2}{r} + \frac{u}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial y} - \nu\frac{\partial^2 w}{\partial y^2} = -\frac{U^2}{r} \tag{18b}$$

The equation of continuity for the boundary-layer flow remains unchanged;

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} + \frac{w}{r} + \frac{\partial v}{\partial y} = 0 \tag{1c}$$

The boundary conditions now are

$$\left.\begin{array}{l} u = w = v = 0 \quad \text{for} \quad y = 0 \\ \left.\begin{array}{l} u \rightarrow U \\ w \rightarrow 0 \end{array}\right\} \text{as} \quad y \rightarrow \infty \end{array}\right\} \tag{18c}$$

For main flows such that $W = 0$, u and w are defined as follows:

$$u \equiv UF'(\eta) \tag{19a}$$

$$w \equiv \overline{W}G'(\eta) \tag{19b}$$

where $\overline{W} = \overline{W}(r,\theta) \neq 0$. The boundary conditions on F' and G' required to satisfy boundary conditions on u and w in equations (18) $(W = 0,\ \overline{W}(r,\theta) \neq 0)$ are

$$\left.\begin{array}{l} F'(0) = G'(0) = 0 \\ \lim\limits_{\eta \rightarrow \infty} F'(\eta) = 1 \\ \lim\limits_{\eta \rightarrow \infty} G'(\eta) = 0 \end{array}\right\} \tag{20a}$$

The expression for v obtained by integration of the continuity equation (1c) is the same as equation (5) with W being replaced by $\overline{W}$;

$$v = -\frac{\sqrt{\nu}}{g}\left[\left(\frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}\right)F + \left(\frac{\partial \overline{W}}{\partial r} - \overline{W}\frac{\partial \ln g}{\partial r} + \frac{\overline{W}}{r}\right)G\right] - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}yF' - \overline{W}\frac{\partial \ln g}{\partial r}yG' + f(r,\theta) \tag{21}$$

As before, the boundary conditions chosen as sufficient to provide that $v = 0$ for $y = 0$ are

$$\left.\begin{aligned} F(0) = G(0) &= 0 \\ f(r,\theta) &= 0 \end{aligned}\right\} \tag{20b}$$

Substitution of equations (19) and (21) into equations (18a) and (18b) produces

$$\frac{\overline{W}}{r}(F'G' - GF'') + \frac{1}{r}\frac{\partial U}{\partial \theta}(F'^2 - FF'' - 1) + \overline{W}\frac{\partial \ln U}{\partial r}F'G' + \frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FF'' +$$

$$\frac{\overline{W}}{2}\frac{\partial \ln g^2}{\partial r}GF'' - \frac{\partial \overline{W}}{\partial r}GF'' - g^2F''' = 0 \tag{22}$$

$$\frac{U^2}{\overline{W}r}(1 - F'^2) + \frac{U}{r}\frac{\partial \ln \overline{W}}{\partial \theta}F'G' + \frac{\partial \overline{W}}{\partial r}(G'^2 - GG'') - \frac{1}{r}\frac{\partial U}{\partial \theta}FG'' +$$

$$\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FG'' + \frac{\overline{W}}{2}\frac{\partial \ln g^2}{\partial r}GG'' - \frac{\overline{W}}{r}GG'' - g^2G''' = 0 \tag{23}$$

The argument concerning determination of the o.d.e. conditions by means of relations between the coefficients of the functions of η in equations (22) and (23) remains unchanged. These coefficients, it may be noted, are the same as the coefficients for equations (6) and (7) with W replaced by $\overline{W}$. Thus, the expressions for $\overline{W}$, U, and g^2 can be written

$$\left.\begin{aligned} \overline{W}(r,\theta) &= br^n e^{m\theta} \quad \text{for} \quad b \neq 0 \\ W &= 0 \\ U(r,\theta) &= ar^n e^{m\theta} \\ g^2(r,\theta) &= cr^{n-1}e^{m\theta} \end{aligned}\right\} \tag{24}$$

The resulting ordinary differential equations are $(W = 0)$

$$b(n + 1)\, F'G' + am\left(F'^2 - \frac{FF''}{2} - 1\right) - \frac{b(n + 3)}{2}GF'' - cF''' = 0 \tag{25}$$

$$\frac{a^2}{b}(1 - F'^2) + am\left(F'G' - \frac{FG''}{2}\right) + bnG'^2 - \frac{b(n + 3)}{2}GG'' - cG''' = 0$$

$$\tag{26}$$

RESULTS AND DISCUSSION

The analysis of three-dimensional, laminar, incompressible, thin-boundary-layer flows having similarity with respect to polar coordinates has led to solutions for mainstream flows described by equation (14) or (24). As a result of this analysis, table I has been prepared, which summarizes the two cases of mainstream flows over a flat or slightly curved surface for which the boundary-layer flows have the required similarity.

4177

As described earlier, secondary flows exist even when the radial component of mainstream flow vanishes; $W = 0$. For such cases a function $\overline{W} = \overline{W}(r,\theta) \neq 0$ is defined and the boundary-layer radial component of flow is expressed as

$$w = \overline{W}G'(\eta) \tag{19b}$$

The table presents these two cases $W \neq 0$ and $W = 0$, $\overline{W} \neq 0$, for the corresponding forms of the tangential component $U = U(r,\theta)$, which permits a solution by reduction of the boundary-layer equations to ordinary differential equations.

Mainstream

When $W \neq 0$, the mainstreams are spiral flows. For $W = 0$, circular mainstream flows are obtained.

$U,W,\overline{W}$. - In regions where the thin-boundary-layer theory is applicable, the mainstream is very nearly parallel to the surface; U and W are functions of r and θ only.

The analysis shows that for the similarity solutions considered here only one form of U and W (or $\overline{W}$) is possible; that is,

$$U = ar^n e^{m\theta}$$

$$W \text{ (or } \overline{W}) = br^n e^{m\theta}$$

When $W = 0$, by choosing $m = 0$ (case I), many of the most frequently encountered flow distributions can be obtained by suitable choices of n. Corresponding to $n = -1$ is free-vortex flow; for $n = 1$, wheel-type flow. In all cases, W/U or $\overline{W}/U$ are constants.

Projection of mainstream on surface. - The equation for the projection of the mainstream on the surface may be obtained by integrating

$$\frac{W}{U} = \frac{dr}{r\,d\theta} \tag{27a}$$

for each case yielding

$$r = c_6 e^{\frac{b}{a}\theta} \tag{27b}$$

The slope of the projected streamline with respect to $\theta = 0$ obtained from

$$\text{slope} = \frac{r + \frac{dr}{d\theta}\tan\theta}{\frac{dr}{d\theta} - r\tan\theta} = \frac{\frac{b}{a}\tan\theta + 1}{\frac{b}{a} - \tan\theta} \tag{28}$$

is found to be independent of radial position r.

Irrotationality. - For the mainstream flows considered in this investigation and in regions of thin boundary layers (as assumed for the analysis), only the component of vorticity normal to the surface

$$\frac{1}{r}\frac{\partial W}{\partial \theta} - \frac{\partial U}{\partial r} - \frac{U}{r}$$

can be much different from zero (ref. 11). The values of the constants specified under the chart listing "Irrotationality" were obtained in each case from

$$\frac{1}{r}\frac{\partial W}{\partial \theta} - \frac{\partial U}{\partial r} - \frac{U}{r} = (mb - an - a)\, r^{n-1} e^{m\theta} \underset{\cdot}{=} 0 \tag{29}$$

These values serve to set the conditions for nearly irrotational mainstream flows.

Boundary Layer

As discussed in reference 11, the physical interpretation of the boundary-layer behavior that the mathematical representations purport to describe is best found by examining the behavior of η and, in particular, $g(r,\theta)$. The boundary-layer thickness was shown (ref. 11) to be inversely proportional to g. In order for the theoretical boundary layer to have a beginning at a leading edge, as in a real fluid, there should, therefore, be a line along the surface for which $g(r,\theta)$ is infinite.

In the solutions presented here, this occurs in the finite part of the plane only at the point $r = 0$ for values of $n < 1$. For $n > 1$, the boundary layer may be considered to have a beginning only at $r = \infty$; there the mainstream velocities take on "infinite" values.

Ordinary Differential Equations

The actual numerical solutions of the ordinary differential equations are beyond the scope of the present investigation. The literature contains examples of numerical solutions that have been calculated for particular values of the constants. Some of these examples are noted in the listing "Comments and references" associated with each case in the chart.

The present analysis then serves only to display the ordinary differential equations that can be obtained with the underlying assumptions. In any particular case of interest for which the equations are appropriate, the existence of the numerical solution and its computation must be obtained individually. Nevertheless, some general remarks can be made here (as in ref. 11) concerning the numerical solutions.

Separation of F and G. - Under certain choices of the free constants involved, the functions F and G are separable; that is, one equation of the pair of ordinary differential equations will contain terms in only one of these functions and its derivatives. Numerical solutions are much more readily obtained in such cases than when the functions are not separated. It can be noted from the table that by choosing $a = 0$ in case I, equation (17) contains only terms in G and its derivatives. This corresponds to mainstreams having no tangential components of flow. Equation (17) becomes a Falkner-Skan type equation with known solutions (refs. 1 and 2). Although it is not apparent from the equations in the table alone, when $a = 0$, in case I, then $u = 0$ (ref. 12), equation (1a) disappears, and so does equation (16). Such flows are really two-dimensional flows originating from a stagnation point and flowing out along straight radial lines.

Linearity in u or w. - As discussed in reference 11 and applied in reference 10, an extension of the solutions beyond strict similarity of the velocity component can sometimes be made by addition of solutions where the boundary-layer equations are linear in u or w. Such extensions are not possible for the boundary-layer flows investigated here, because equation (1b) is always nonlinear in u and in w except for the trivial case of no mainstream flow.

4177

CONCLUDING REMARKS

Solutions are obtained for the mainstream flow patterns for boundary-layer flows having classical similarity with respect to stationary polar coordinates. The results are summarized in the table. The exact solutions obtained are beset with the difficulty that their boundary layers have no proper leading edge in the finite part of the plane, whereas in turbomachines a definite leading edge is generally required. Nevertheless, the analysis enables a study of the properties of the boundary-layer flows and may have direct applicability when attention is confined to appropriate regions of the flow.

The solutions are considered completed for the sake of this investigation when the boundary-layer equations have been transformed into ordinary differential equations. The actual numerical solutions for the ordinary differential equations so derived are beyond the scope of this work.

There are three configurations of main-flow streamlines for which the similarity solutions here could be obtained. The mainstream may be (1) a stagnation-type flow out along radial lines from a stagnation point, (2) spiral flow out from (or in toward) a central point, or (3) circular flow. By suitable choice of the free constants involved, the flows may include at will cases of acceleration or deceleration in the radial or tangential directions. The solutions obtained here thus constitute an extension of the similarity solutions obtained in reference 11 for rectangular coordinate systems.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, July 24, 1956

REFERENCES

1. Falkner, V. M., and Skan, Sylvia W.: Some Approximate Solutions of the Boundary Layer Equations. R. & M. No. 1314, British A.R.C., Apr. 1930.

2. Hartree, D. R.: On an Equation Occurring in Falkner and Skan's Approximate Treatment of the Equations of the Boundary Layer. Proc. Cambridge Phil. Soc., pt. 2, vol. 33, Apr. 1937, pp. 223-239.

3. Mangler, W.: Boundary Layers on Bodies of Revolution in Symmetrical Flow. Reps. and Trans. No. 55, GDC/689T, British M.A.P., Apr. 15, 1946.

4. Cooke, J. C.: The Boundary Layer of a Class of Infinite Yawed Cylinders. Proc. Cambridge Phil. Soc., pt. 4, vol. 46, Oct. 1950, pp. 645-648.

5. Howarth, L.: The Boundary Layer in Three Dimensional Flow. II - The Flow Near a Stagnation Point. Phil. Mag., ser. 7, vol. 42, 1951, pp. 1433-1440.

6. Mager, Artur, and Hansen, Arthur G.: Laminar Boundary Layer over Flat Plate in a Flow Having Circular Streamlines. NACA TN 2658, 1952.

7. Sowerby, L.: Secondary Flow in a Boundary Layer. Rep. No. AERO 2512, British R.A.E., Mar. 1954.

8. Mager, Artur: Three-Dimensional Laminar Boundary Layer with Small Cross-Flow. Jour. Aero. Sci., vol. 21, no. 12, Dec. 1954, pp. 835-845. 4177

9. Loos, Henk G.: A Simple Laminar Boundary Layer with Secondary Flow. Jour. Aero. Sci., vol. 22, no. 1, Jan. 1955, pp. 35-40.

10. Hansen, Arthur G., and Herzig, Howard Z.: Cross Flows in Laminar Incompressible Boundary Layers. NACA TN 3651, 1956.

11. Hansen, Arthur G., and Herzig, Howard Z.: On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. I - Similarity with Respect to Stationary Rectangular Coordinates. NACA TN 3768, 1956.

12. Hayes, Wallace D.: The Three-Dimensional Boundary Layer. NAVORD Rep. 1313, NOTS 384, U.S. Naval Ord. Test Station, May 9, 1951. (Bur. Ord. Task Assignment NOTS-36-Re3d-441-3.)

13. Schlichting, Hermann: Boundary Layer Theory. McGraw-Hill Book Co., Inc., 1955.

14. Bödewadt, U. T.: Die Drehströmung über festem Grunde. Z.a.M.M., Bd. 20, Heft 5, Oct. 1940, pp. 241-253.

TABLE I. - SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES

CASE I

$W \neq 0$

U	$ar^n e^{m\theta}$
W	$br^n e^{m\theta}$, $b \neq 0$
η	$y\left(\frac{cr^{n-1}e^{m\theta}}{\nu}\right)^{1/2} = y\left(\frac{cU}{\nu ar}\right)^{1/2} = y\left(\frac{cW}{\nu br}\right)^{1/2}$, $c \neq 0$
Ordinary differential equations	(16) $b(n + 1)(F'G' - 1) + am\left(F'^2 - \frac{FF''}{2} - 1\right) - \frac{b(n + 3)}{2} GF'' - cF''' = 0$ (17) $\frac{a^2}{b}(1 - F'^2) + am\left(F'G' - \frac{FG''}{2} - 1\right) + bn(G'^2 - 1) - \frac{b(n + 3)}{2} GG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0$; $\lim_{\eta \to \infty} F'(\eta) = \lim_{\eta \to \infty} G'(\eta) = 1$
Projection of mainstream on surface	$r = Ce^{\frac{b}{a}\theta}$, spiral flow streamlines, $(a \neq 0)$
Irrotationality	$bm - an - a = 0$
Linearity in u Linearity in w	(1a) $m = 0$ (1a) Linear (1b) Always nonlinear
Separation of F and G	(16) Not possible for this case, $(b \neq 0)$ (17) $a = 0$
Comments and references	$a = 0$, stagnation flow. Eq. (16) vanishes. Eq. (17) becomes a Falkner-Skan equation, which is completely solved in refs. 1 and 2. Ref. 3: $a = m = 0$, $b = c/2$, $n = 1$. Ref. 13, p. 71: plane stagnation flow: $a = m = 0$, $n = 3$, $b = 1/3$, $c = 1$, $G = \varphi$ (ref. 13). Ref. 13, p. 74: Three-dimensional stagnation point flow, axisymmetrical case: $a = m = 0$, $n = 1$, $b = c$, $G = \varphi$ (ref. 13).

4177

TABLE I. - Concluded. SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES

CASE II

$W = 0, \overline{W} \neq 0$

U	$ar^n e^{m\theta}$
W	0
η	$y\left(\frac{cr^{n-1}e^{m\theta}}{\nu}\right)^{1/2} = y\left(\frac{cU}{\nu ar}\right)^{1/2}, \ c \neq 0$
Ordinary differential equations	(25) $b(n + 1)(F'G') + am\left(F'^2 - \frac{FF''}{2} - 1\right) - \frac{b(n + 3)}{2}GF'' - cF''' = 0$ (26) $\frac{a^2}{b}(1 - F'^2) + am\left(F'G' - \frac{FG''}{2}\right) + bn\left(G'^2\right) - \frac{b(n + 3)}{2}GG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0; \ \lim_{\eta\to\infty} F'(\eta) = 1; \ \lim_{\eta\to\infty} G'(\eta) = 0$
Projection of mainstream on surface	r = C, circular flow streamlines
Irrotation-ality	$a(n + 1) = 0$
Linearity in u Linearity in w	(1a) $m = 0$ (1a) Linear (1b) Always nonlinear
Separation of F and G	(25) (26) Not possible for this case $a = 0$
Comments and references	$a = 0$, no flow Ref. 13, p. 157, eq. (10.9), (discussing work of ref. 14): $a = b = c$, $m = 0$, $n = 1$. F' here = G (ref. 13). G' here = F (ref. 13)